1D model for the dynamics and expansion of elongated Bose-Einstein condensates

Pietro Massignan and Michele Modugno

INFM - LENS - Dipartimento di Fisica, University of Florence
Via Nello Carrara 1, 50019 Sesto Fiorentino, Italy


Nordita
Kobenhavn, 6 March 2003
Periodic potentials are powerful tools to investigate coherence properties.

Via the dipole force exerted by an off-resonant laser standing wave it is possible to produce an almost perfect and infinite periodic potential:

\[ U_{1D}(r_z) = s \cdot E_r \cdot \cos^2 \left( \frac{2\pi r_z}{\lambda_{opt}} \right) \]

(recoil energy: \( E_r \equiv \frac{\hbar^2}{2m\lambda_{opt}^2} \))

- pulsed atom laser
- condensates \( \oplus \) lattices \( \rightarrow \) superfluidity and Josephson effects
- matter diffraction

¿Qubits arrays?
Why a 1D model?

* BECs at $T = 0$ are usually described within the framework given by the Gross-Pitaevskii Equation (1961):

$$i\hbar \frac{\partial}{\partial t} \Psi(\mathbf{r}, t) = \left[ -\frac{\hbar^2}{2m} \nabla^2 + U(\mathbf{r}, t) + gN |\Psi(\mathbf{r}, t)|^2 \right] \Psi(\mathbf{r}, t)$$

$$g \equiv \frac{4\pi \hbar^2 a}{m} \quad (a: \text{scattering length})$$

* The absence of analytical solutions often implies a numerical approach

but

rapid spatial potential variations $\implies$ heavy numerical simulations

↓

uni-dimensional geometry:

**cigar-shaped condensates** $\implies$ need for a 1D model

and axial dynamics

$$1D: \quad i\hbar \frac{\partial}{\partial t} \psi(z, t) = \left\{ -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial z^2} + U(z, t) + ? \right\} \psi(z, t)$$
3 recent proposals for the dimensionality reduction

**Statiscal renormalization methods**

**Weak auto-interaction**  Jackson, Kavoulakis and Pethick (98)

\[ \frac{N\alpha}{a\bar{\omega}} \ll 1 : \]

\[ i\hbar \frac{\partial}{\partial t} \psi(z, t) = \left\{ -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial z^2} + U(z) + \frac{gN}{\pi} <R^2> |\psi(z, t)|^2 \right\} \psi(z, t) \]

\[(a_\perp \equiv \sqrt{\frac{\hbar}{m\omega_\perp}}, \quad <R^2>=<X^2>+<Y^2>=2a^2_\perp)\]

**Strong auto-interaction**  Trippenbach, Band and Julienne (00)

(Thomas-Fermi limit)

\[ \frac{N\alpha}{a\bar{\omega}} \gg 1 : \]

\[ i\hbar \frac{\partial}{\partial t} \Psi(r, t) = \left\{ \sum_{i=1}^{d}\left( -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x_i^2} + \frac{m}{2}\omega_i^2 x_i^2 \right) + G^{(d)}_0 |\Psi(r, t)|^2 \right\} \Psi(r, t) \]

with

\[ G^{(d)}_0 = G^{(d)}_0(N, a, \bar{\omega}) \]
Dynamical renormalization method

\[ \text{III } \] Factorized wave-function, gaussian transverse part\(^a\):

\[
\text{Ansatz: } \left\{ \begin{align*}
\Psi(r, t) &= \phi(r, t; \sigma_o(z, t)) \psi(z, t) \\
\phi(r, t; \sigma_o(z, t)) &= \frac{1}{\sqrt{\pi \sigma_o}} e^{-r^2/2\sigma_o^2}
\end{align*} \right.
\]

\[ S[\Psi] = \int dt \int dz \ d^2r \ \Psi^* \left\{ i\hbar \partial_t + \frac{\hbar^2}{2m} \nabla^2 - \frac{1}{2} m \omega_z^2 r^2 - U(z, t) - \frac{gN}{2} |\Psi|^2 \right\} \Psi \]

Inserting the Ansatz in the action of the system, under the "slowly varying approximation" \( \nabla^2 \phi \approx \nabla_z^2 \phi \) one gets a 1D dynamical equation for the axial wave-function (NPSE):

\[ i\hbar \frac{\partial}{\partial t} \psi(z, t) = \left\{ -\frac{\hbar^2}{2m} \nabla_z^2 + U(z, t) + \frac{gN}{2\pi \sigma_o^2} |\psi(z, t)|^2 + \frac{\hbar \omega_z}{2} \left( \frac{a_\perp^2}{\sigma_o^2} + \frac{\sigma_o^2}{a_\perp^2} \right) \right\} \psi(z, t) \]

where the transverse width is given by: \( \sigma_o(z, t) = a_\perp \sqrt{1 + 2aN |\psi(z, t)|^2} \)

\[ \text{The NPSE (III) gives an axial description of a condensate in a time-independent harmonic trap much more accurate than (I) and (II).} \]

\(^a\)L. Salasnich, Laser Phys. 12, 198 (2002);

(I), (II) and (III) could be used to describe the ground state but not the free expansion of a condensate:

✘ (I), (II): for a *cigar-shaped* condensate \((\omega_\perp \gg \omega_z)\) in the TF limit a generic (statically renormalized) 1D-GPE *largely* overestimates the axial width of the freely expanding wave-packet (see Fig. 5, coming soon);

✘ (III) is derived in presence of a constant non-zero transverse harmonic confinement:

\[
U_{\perp}^{ho} \to 0 \Rightarrow a_{\perp} \to \infty \Rightarrow \sigma_o \to \infty.
\]

✔ The interplay between the axial and radial dynamics is necessary to account for the quadrupole oscillations.

✔ Solution

\[
\begin{align*}
\text{coordinates rescaling} & \quad \text{gaussian transverse hypothesis} \\
\text{gauge transform} & \quad \leadsto \text{dr-GPE}
\end{align*}
\]
Introduction of rescaled coordinates: $x \equiv \frac{r}{\lambda}$

Local gauge transformation: $
\Psi(r, t) = e^{\frac{i m}{2 \hbar} \sum r_j^2 \frac{\dot{x}_j}{x_j}} \frac{\tilde{\Psi}(x, t)}{\sqrt{\lambda_x(t) \lambda_y(t) \lambda_z(t)}}$

\[
\left\{ \begin{array}{l}
\frac{\ddot{x}_j(t)}{\lambda_j(t)} + \omega_j^2(t) = \frac{\omega_j^2(0)}{\lambda_j^2(t) \lambda_x(t) \lambda_y(t) \lambda_z(t)} \\
\lambda(0) = 1, \quad \dot{\lambda}(0) = 0 \implies \tilde{\Psi}(x, 0) \equiv \Psi(r, 0)
\end{array} \right.
\]

\[
GPE \rightarrow i\hbar \partial_t \tilde{\Psi}(x, t) = \left\{ \ldots + \frac{m}{2} \sum_j \lambda_j^2 x_j^2 \left( \frac{\dddot{x}_j(t)}{\lambda_j(t)} + \omega_j^2(t) \right) + \ldots \right\} \tilde{\Psi}(x, t)
\]

\[
i\hbar \partial_t \tilde{\Psi}(x, t) = \left\{ - \frac{\hbar^2}{2m} \sum_j \frac{\partial^2}{\partial x_j^2} \frac{\lambda_j^2(t)}{\lambda_j^2(t)} \right\} + \frac{U_{ho}(x, 0) + gN|\tilde{\Psi}(x, t)|^2}{\lambda_x(t) \lambda_y(t) \lambda_z(t)} \tilde{\Psi}(x, t)
\]

\[\iff \text{total elimination of the harmonic potential temporal dependency}\]

Since the performed transformation is unitary, the latter equation is exact.

\[U_{ho}(t) \sim U_{ho}(0) \iff \text{the rescaled wave-function } \tilde{\Psi} \text{ evolves in a fictitious harmonic potential, whose characteristic lengths are fixed to their } t = 0 \text{ values}\]

---

Y. Castin and R. Dum, Phys. Rev. Lett. 77, 5315 (1996);

Assuming a cylindrically-symmetric potential, sum of a time-dependent harmonic term and an additional axial component:

\[ U(r, t) = \frac{m}{2} \sum_{j=z, \perp} \omega_j^2(t) r_j^2 + U_{1D}(r_z, t) \]

we impose the gaussian factorization on the rescaled wave-function:

\[
\begin{align*}
\tilde{\Psi}(x, t) &= \tilde{\phi}(x, y, t; \sigma(z, t)) \tilde{\psi}(z, t) \\
\text{Ansatz:} & \left\{ \begin{array}{l} \\
\tilde{\phi}(x, y, t; \sigma(z, t)) = \frac{1}{\sqrt{\pi} \sigma} e^{-\frac{x^2+y^2}{2\sigma^2}} \\
\end{array} \right.
\end{align*}
\]

Under the hypothesis \( \nabla^2 \phi \approx \nabla^2_{\perp} \phi \), by variational deduction one obtains the dynamical equation for the axial wave-function (\textbf{dr-GPE}):}

\[
\begin{align*}
\frac{i\hbar}{\partial t} \tilde{\psi}(z) = & \left\{ -\frac{\hbar^2}{2m} \frac{\nabla^2_{\perp}}{\lambda_{\perp}^2} + \frac{\hbar^2}{2m} \frac{1}{\lambda_z^2 \sigma^2} + U_{1D}(\lambda_z z, t) + \\
& \frac{1}{\lambda_z \lambda_{\perp}^2} \left[ \frac{m \omega_z^2(0)}{2} z^2 + \frac{m \omega_{\perp}^2(0)}{2} \sigma^2 + \frac{gN}{\sqrt{2 \pi} \sigma^2} |\tilde{\psi}|^2 \right] \right\} \tilde{\psi}(z)
\end{align*}
\]

where the transverse width is given by:

\[
\sigma(z, t) = a_{\perp}^{o} \sqrt{\lambda_{z}(t) + 2aN|\tilde{\psi}(z, t)|^2}
\]

\[
\left( a_{\perp}^{o} \equiv \sqrt{\hbar/m \omega_{\perp}(0)} \right)
\]

The transverse width of the true wave function \( |\phi|^2 \propto e^{-r^2/\Sigma^2} \) is given by \( \Sigma(r_z, t) \equiv \sigma \cdot \lambda_{\perp} \)
Properties of the $dr$-GPE

$dinamically$ $rescaled$-$GPE$ : \[
\begin{align*}
1D \text{ non-linear Schrodinger eq. (} & \sim \sim \text{ GPE)} \\
\text{function of the rescaled coordinate } & \ z(t) \\
\text{with constant harmonic potential}
\end{align*}
\]

☞ The $dr$-GPE is energy conserving and reduces to the NPSE in case of a time-independent harmonic potential.

☞ The variational parameter $\sigma$ allows the model for an intrinsic description of the radial dynamics of the system.

☞ The evolution of $\tilde{\Psi}$ due to the harmonic confinement variations is mostly absorbed by the $\langle \ldots \rangle$ transformations (in the TF limit $|\tilde{\Psi}(t)| = |\tilde{\Psi}(0)|$).

☞ The fictitious constant harmonic potential gives sense to a gaussian factorization even in the case of a sudden release of the external confinement.

☞ Since the numerical solution of the $\lambda$ equations is straightforward, the propagation of the $dr$-GPE requires the same computational effort of a simple 1D-GPE.
Figure 1: Radial size of the ground state in an harmonic potential as a function of N. The shift between the two curves is expected, since the TF approximation systematically underestimates the condensate radii (especially at low N).

Figure 2: Radial density integrated over the axial coordinate of a condensate in the ground state of an harmonic potential.

We use typical LENS parameters: $2 \cdot 10^4 < N < 2 \cdot 10^5$ Rb atoms in cigar-shaped configurations ($\nu_c = 9$ Hz, $\nu_\perp = 92$ Hz) exposed to lattices with $\lambda_{opt} = 795$ nm and $0 \leq s \leq 6$. 
An interesting feature of this model is the possibility to describe oscillations induced by modulations of the axial or the radial part of the trapping potential.

Figure 3: Evolution of the axial (top) and radial (bottom) sizes of a condensate performing shape oscillations, obtained suddenly weakening the radial confinement:

\[ \nu_\perp : 92 \, Hz \rightarrow 80 \, Hz. \]

In both graphs it is possible to observe the superposition of the quadrupole and the faster transverse breathing frequencies (\( \omega^Q = \sqrt{5/2} \omega_z, \omega^{TB} = 2 \omega_\perp \)).
Figure 4: Schrodinger, \( dr \)-GPE, 1D-TF predictions and experimental values for the axial density of the condensate after a free expansion of \( t_{\text{exp}} = 29.5 \) ms.

\[
\text{TF } dr\text{-GPE: } i\hbar \partial_t \tilde{\psi}(z) = \left\{ \ldots + \frac{1}{\lambda_x \lambda_y} \left[ \frac{3}{2} \hbar \omega (0) \sqrt{2aN} |\tilde{\psi}| \right] \right\} \tilde{\psi}(z)
\]

Figure 5: \( dr \)-GPE, 1D-TF and 3D-TF predictions for the axial and radial sizes of the condensate during a free expansion from an harmonic potential.
Even when a strong lattice causes deep modulations of $|\Psi|^2$, the $\nabla^2 \phi \approx \nabla_{\perp}^2 \phi$ approximation is still a good one: the $dr$-GPE results are really close to those of the complete equation.

Figure 6: Axial density $\rho(z)$ (top) and rms transverse radius, averaged only on the transverse coordinates and plotted as a function of the axial coordinate (bottom); the insets show a magnification of the central region (lattice intensity $s = 5$).
Free expansion from a combined (harmonic+optical) potential

Figure 7: Axial density of a coherent array of condensates, initially trapped in an $s = 5$ optical lattice, imaged after a free expansion of 29.5 ms.

**Lattice on: collective modes**

The presence of an optical lattice renormalizes the normal mode frequencies:

$$\omega_z^D \rightarrow \sqrt{m/m^*} \omega_z^D$$

Figure 8: Effective mass $m^*$ as a function of the lattice intensity compared to that of a single particle in a periodic potential. Inset: the frequency of the dipole mode is compared with a recent LENS experiment (the dotted line is their theoretical curve).

---


Figure 9: Structure of the lowest two bands for a single particle in a 1D periodic potential with $s = 2$ (solid), compared to the free particle case (dashed). Inset: effective mass (inversely proportional to the band curvature) in the first band.

Figure 10: Left: axial width of a condensate adiabatically loaded in a 1D lattice ($s = 2$) for different initial velocities (solid: $q = 0$, dotted: $q = 0.685$), plotted as a function of time. Inset: lattice intensity ramp of the thought experiment.
Right: aspect ratio after 25 ms of expansion, plotted as a function of the initial BEC velocity.
**dr-GPE**: 1D effective equation able to describe ground state and dynamics of BECs confined in *cigar-shaped* harmonic traps, generically time-dependent and containing an arbitrary axial component.

Applications:
- collective oscillations and excitations
- exploration of new dynamical effects, band spectroscopy
- behaviour in generic 1D potentials (gravity, lattices, barriers)